Combinatorial polytopes and intersection homology

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The talk will explain why, conjecturally, this matrix gives the effective J in dimension 5. The talk also explains why locality makes the starred entry especially interesting.

$$M_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1^* \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

M_5 is an 8×8 matrix. (Fibonacci numbers in red in this talk.)

Conjecturally, by using brute force we know M_i for $i \leq 5$, but don't yet have even a guess for M_6 (about $2^{13 \times 12/2}$ candidates) and beyond.

In a nutshell (to be repeated)

- Simplex S_D is all probability distributions on finite set D.
- Convex polytope X is any image of simplex under a linear map.
- Bonus topic: Quantum entanglement and Bell polytopes.
- Combinatorics constructs many interesting convex polytopes.
- Flag vector gives map $X \mapsto [X]$ to a linear space $\mathcal{F} = \mathcal{F}_{\dim X}$.
- The dimension of $\mathcal{F}$ is a Fibonacci number, number of elements in $\mathcal{W}$.
- Want to find a special basis e_i (for $i \in \mathcal{W}$) of $\mathcal{F}$ with ...
- ... $g_i(X) \geq 0$ when $[X] = \sum_{i \in \mathcal{W}} g_i(X) e_i$, for X any convex polytope.
- Already have CD basis $\tilde{e}_i$ for $\mathcal{F}$. Write $[X] = \sum_{i \in \mathcal{W}} \tilde{g}_i(X) \tilde{e}_i$.
- ... $g_i = \sum_{j \in J} \tilde{g}_j$ for some subset $J = J(i)$ of $\mathcal{W}$.
- ... $g_i(X)$ is dim'n of primitive intersection homology space $P^i(X)$.
- ... and this is how to find the $P^i(X)$, and so prove $g_i(X) \geq 0$.
- **Problem:** Find, or guess, the subsets $J(i)$ of $\mathcal{W}$. (Eg M_5 for $d = 5$.)

. . . a convex polytope?

- finite intersection of half spaces.
- convex hull of finite set of points.
- image of simplex under linear map (best for this talk).
- linear subspace of $\mathbb{R}^d$ (or of $\mathbb{R}^D$, for D a finite set).

. . . a combinatorial (convex) polytope?

- simplex, cube, or perhaps higher-order subset polytope.
- graph edge polytope, subgroup polytope, Fibonacci polytope, etc.
- associahedron, permutahedron, etc.

. . . intersection homology?

- complete and pure extension of homology of simple polytopes . . .
- . . . to general polytopes (and algebraic varieties).

What is intersection homology? (to be repeated)

Homology of a simple convex polytope

- Facet-translation volume-polynomial ...
- ... turned into a graded ring $H = \bigoplus_0^d H^i$.
- With $H^0 = \mathbb{R}$ (scalar functions) and $H^d \cong \mathbb{R}$ (volume forms).
- With a 'hyperplane class' $\omega : H^i \to H^{i+1}$.

And H is **pure**, which means:

- Poincaré duality: $H^i \otimes H^{d-i} \to H^d$ is vector space duality.
- Hard Lefschetz: $\omega^j : H^i \to H^{i+j}$ is isomorphism (for $2i + j = d$).
- $h_i = \dim H^i$ is a linear function of the face (or flag) vector.

Intersection homology (IH) of general convex polytope

- Pure extension of homology to general convex polytopes.
- Encodes whole of the flag vector (once all of IH has been found).

What is quantum entanglement?

History Einstein resisted the usual theory of quantum mechanics. With Podolsky and Rosen he developed the EPR paradox (1935). This is an experiment which, they said, shows that this theory if complete implies a 'spooky action at a distance' that propagates faster than light.

The EPR experiment Create two particles, A and B, that share state and become distant. Now measure the momentum of A and of B when neither is in the light cone of the other. The two momenta will sum to zero.

Reality All parties (and experiment) agree on what happens.

The paradox EPR argues that, on the usual theory, measuring the momentum of A **causes** B to have the opposite momentum. Spooky.

EPR argues on the basis of a physical insight (local realism – next slide) that the EPR experiment shows that the usual theory is incomplete.

Respect physical insight because it was a major component of Einstein's discovery of special and general relativity.

Local realism, Bell's theorem and Bell's polytope

Local realism is the basis of the EPR paradox. The words mean

- **local:** distant events don't affect behaviour (c is the speed limit)
- **realism:** any measurement on a system, whose value can be predicted, corresponds to an element of the system's physical reality

John Bell (1964) showed that local realism is inconsistent with quantum theory, and described an experiment that could distinguish the two.

Bell's experiment involves correlations between EPR-style observations. Local realism implies that these correlations are given by *linear inequalities*, and so define the *Bell polytope* (for the given system).

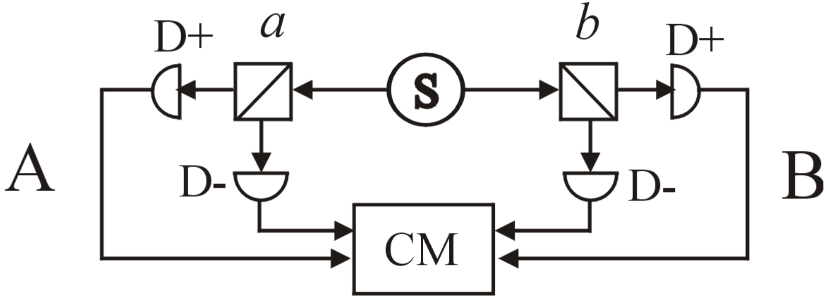
Quantum theory gives a rounded correlation region that contains the Bell polytope (whose vertices lie on the boundary of the quantum region?).

Bell-type experiments by Freedman and Clauser (1972) and later many others have confirmed quantum mechanics (and refuted local realism).

Without EPR when would we have refuted local realism?

Scheme of a CHSH two-channel Bell experiment

CHSH stands for Clauser, Horne, Shimony and Holt (1969). Aspect's 1982 experiment was of this type.



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Description of CHSH Bell-type experiment

Quantum fluctuations, for example, can be used to produce photon pairs that share state (are entangled). For each pair, send out its two photons in opposite directions A and B (diagram on previous page).

On the A side there is a polarisation detector set at angle a. It gives either $+$ or $-$. Similarly with B at angle b. There are four possible outcomes, $++, +-, -+, --$. Repeat M times to obtain N_{++}, N_{+-} etc.

Definition (Correlation fraction)

For each pair (a, b) of filter angles, $E(a, b)$ measures correlation, where

$$E(a, b) = (N_{++} + N_{--} - N_{+-} - N_{-+})/M$$

Local realism implies (using $a = 0, a' = \frac{\pi}{4}, b = \frac{\pi}{8}, b' = \frac{3\pi}{8}$) that

$$-2 \leq E(a, b) - E(a, b') + E(a', b) + E(a', b') \leq 2$$

Such inequalities define the *Bell polytope* for the experiment.

Combinatorial polytopes

Definition

A *combinatorial polytope* is a convex polytope whose vertices (and other faces) are interesting combinatorial objects.

This is a subjective definition. It depends on what you find interesting. Here are some examples (with description of vertex):

- simplex (member of finite set D)
- cube (subset of finite set D)
- associahedron (ordering of associative multiplication)
- graph edge polytope (edge of a graph on given finite set D)
- hypersimplex (has binomial number of vertices)
- Fibonacci polytope (has Fibonacci number of vertices)

We want not only vertices but also other faces to be interesting.

The simplex

The simplex is perhaps the simplest of all convex polytopes (hence the name). In each dimension, of all convex polytopes it has the least number of vertices.

Definition

The *standard* $(d-1)$*-simplex* is the region in $\mathbb{R}^d$ defined by:

- $x_1 + \cdots + x_d = 1$
- $0 \leq x_i \leq 1$

Each point in this simplex represents a probability distribution, where there are d different independent possible outcomes. It has d vertices and d facets. Each vertex corresponds to a basis vector e_i in $\mathbb{R}^d$. Altogether, it has 2^d faces, one for each subset of $\{1, \ldots, d\}$.

Given a finite set D, we can also form *the simplex on* D, denoted by S_D. A vertex of S_D corresponds to a member of D.

The cube

There are many ways to think of a cube:

- The region $0 \leq x_i \leq 1$ for $(x_1, \ldots x_n) \in \mathbb{R}^n$.
- Convex hull of $(\epsilon_1, \ldots \epsilon_n)$, where each ϵ_i is either 0 or 1.
- Convex hull of $\mathbf{e}_I = \sum_{i \in I} \mathbf{e}_i$, as I ranges over the subsets of $\{1, \ldots, n\}$.
- n-fold product I^n of an interval $I = [0, 1]$.

For us, a cube does not have to be 3-dimensional. If it is, we will call it a 3-cube. The 2-cube we call, as usual, a square.

Definition

The *cube* or *subset polytope on a finite set* V is the cube, as defined above, where each vertex of which corresponds to a subset of V.

Note that each vertex of the subset polytope on V is also a face of the simplex on V.

Graph edge polytope - F.(2012)

Let G be a graph on node set N. Each edge of G is a two-element subset of N and so corresponds to a vertex on the cube on N.

Definition

The *graph edge polytope* of a graph G has as vertices the edges of G, thought of as vertices of the cube on the node set N of the graph.

The graph edge polytope on n vertices has dimension at most $n - 1$. Each face is also a graph edge polytope, corresponding to a subgraph of G (but not necessarily vice versa).

Theorem

If a graph edge polytope (on n nodes) has dimension less than the maximum possible (i.e. $\leq n - 2$) then its graph G is disconnected or bipartite (or both).

The proof is a simple exercise.

The hypersimplex

Each element v of V determines a one-element subset $\{v\}$ of V. Thus, or otherwise, it is obvious that the vertices of a simplex are a subset of the vertices of a cube (given by $\sum x_i = 1$).

Recall that the vertices of the graph edge polytope are a subset of the vertices of a cube.

Definition

The r-th *hypersimplex* on a finite set V is the convex polytope whose vertices are the r-element subsets of V (thought of as vertices of the cube on V).

The first hypersimplex (on a finite set V) is (equivalent to) the simplex (also on V). The second hysersimplex (on a finite set V) is the graph edge polytope of the complete graph (also on V).

The r-th hypersimplex is the cube intersected with $\sum x_i = r$.

The Fibonacci polytope [Hsu, Rispoli, Cosares]

The *Fibonacci numbers* $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ are given by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}$$

Let c and d be non-commuting symbols of degree 1 and 2 (so d is double). Then F_n is the number of words in c and d of degree $n-1$. (Too late now to fix this off-by-one error.)

Definition

Let W denote the set of all Fibonacci words of degree d. Each vertex of the *Fibonacci polytope* is the corresponding vertex of the d-cube obtained by replacing c by 0 and d by 10.

The vertices of the Fibonacci polytope are the vertices of the cube that have neither repeated nor trailing 1.

(This definition gives what Rispoli calls the Fibonacci hypercube.)

Zero-one and random polytopes

Definition

A *zero-one convex polytope* is one whose vertices are a subset of those of a standard cube (all of whose vertices have coordinates zero or one).

The cube itself, the simplex, the graph edge polytope, the hypersimplex and the Fibonacci polytope are all examples of zero-one polytopes.

Constructing 'random' convex polytopes is hard. When vertices are chosen continuously at random in $\mathbb{R}^n$, almost always each proper face will be a simplex. Choosing the linear inequalities similarly means each proper link will be a simplex, almost always. So choose from a discrete set of vertices.

Definition

Any zero-one polytope that is interesting will be called *special*.

Pál Turán said that special functions should be called useful functions.

The associahedron [Tamari, Stasheff, Haiman, Lee, ...]

Recall that if multiplication is associative then $a(bc) = (ab)c$, and we can use abc to denote the common value.

Now suppose we wish to compute $abcde$. We can do this in many ways. Here's one way: $(ab)(c(de))$. (How many altogether? A Catalan number.)

Definition

The above is an example of a *complete bracketing* of $abcde$.

Definition

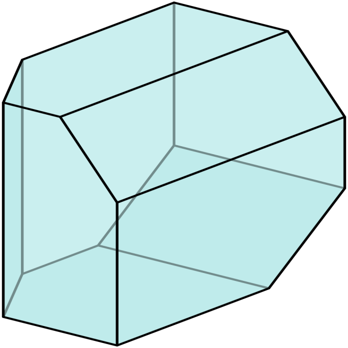
If some (or none) matching brackets are removed from a complete bracketing the result is an *partial bracketing*.

Definition (Tamari, Stasheff)

An *associahedron* (on n terms) is any convex convex polytope whose vertices are the complete bracketings (on n terms) and whose faces are the partial bracketings.

The 3-dimensional associahedron

Five pentagonal faces and three rectangular faces. Each edge is a ‘flip’ $x(yz) = (xy)z$. Each square is a disjoint pair of flips. Each pentagon is bracketings of $wxyz$



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Loday's construction of the associahedron

Tamari (1951) posed the realisation problem. Stasheff (1963) proved that the definition gives a cellular ball. Unpublished constructions by Milnor, Huguet and Tamari, and Haiman. Lee (1989) is first published proof.

Loday (2004) provides a most elegant construction. To each complete bracketing he associates a vector. Here's how, for $((ab)(c(de)))$.

- $n \mapsto (x, y) \mapsto$ smallest subcalculation involving x, y
  $\mapsto$ number of openings on left times closings on right.
- $1 \mapsto (a, b) \mapsto (a \mid b) \mapsto 1 \times 1 = 1$
- $2 \mapsto (b, c) \mapsto ((ab) \mid (c(de))) \mapsto 2 \times 3 = 6$
- $3 \mapsto (c, d) \mapsto (c \mid (de)) \mapsto 1 \times 2 = 2$
- $4 \mapsto (d, e) \mapsto (d \mid e) \mapsto 1 \times 1 = 1.$

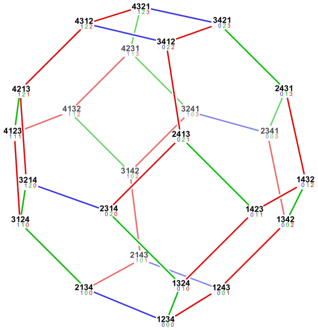
Theorem (Loday (2004))

This construction yields an associahedron.

Q: Why does this work? I don't know.

The permutohedron on four objects

Each vertex is a permutation of $S = (1, 2, 3, 4)$. The faces correspond to strict weak orderings on S.



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- **Problem:** Find, or guess, the subsets $J(i)$ of $\mathcal{W}$. (Eg M_5 for $d = 5$.)

Things to remember (to be repeated)

X is a convex polytope.

$\mathcal{W}$ is Fibonacci words (in C and D).

We already know $\tilde{g}_j(X)$, for all $j \in \mathcal{W}$.

We define $\tilde{g}_J(X) = \sum_{j \in J} \tilde{g}_j(X)$ for $J \subseteq \mathcal{W}$.

We want $g_i(X) = \tilde{g}_J(X)$ for some $J = J(i)$.

We also want $g_i(X) \geq 0$.

If $\tilde{g}_J(X) < 0$ (for some X) then we can't use J.

The h vector for simple polytopes

Definition

A d-dimensional convex polytope X is **simple** if each vertex lies on exactly d edges (or equivalently facets).

The cone on a square (an Egyption pyramid) is not simple. The Cartesian product of two simple polytopes is simple.

Assume X is simple. Put X into $\mathbb{R}^d$, generically. Call the first co-ordinate the height. Each vertex has a different height and so for some i has i downward edges and $d - i$ upward edges (simple plus generic).

Definition (Morse, McMullen)

Each component h_i of the h-vector $(h_0, \ldots, h_d)$ counts how many vertices on X with exactly i downward edges (and $d - i$ upward edges).

Note: Swapping *up* and *down* reverses the h-vector.

The f and g vectors of a simple polytope

Definition

Each component f_i of the f (or face) vector of a simple polytope X counts how many faces of dimension i there are on X.

Theorem (McMullen)

We can compute h from f and vice versa. Therefore $h_i = h_{d-i}$ or, more concisely, h is palindromic. (These are the Dehn-Sommerville equations.)

Proof: Count each face of X at its highest vertex. Then $f = Ch$, where C is an upper triangular matrix of binomial coefficients. Thus, $h = C^{-1}f$ and so is unchanged when we swap up for down.

Definition (McMullen)

Write $g_i = h_i - h_{i-1}$ for $i = 0, \ldots \lfloor d/2 \rfloor$, using $h_{-1} = 0$ so $g_0 = h_0$.

What is intersection homology? (repeat)

Homology of a simple convex polytope

- Facet-translation volume-polynomial ...
- ... turned into a graded ring $H = \bigoplus_0^d H^i$.
- With $H^0 = \mathbb{R}$ (scalar functions) and $H^d \cong \mathbb{R}$ (volume forms).
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And H is **pure**, which means:

- Poincaré duality: $H^i \otimes H^{d-i} \to H^d$ is vector space duality.
- Hard Lefschetz: $\omega^j : H^i \to H^{i+j}$ is isomorphism (for $2i + j = d$).
- $h_i = \dim H^i$ is a linear function of the face (or flag) vector.

Intersection homology (IH) of general convex polytope

- Pure extension of homology to general convex polytopes.
- Encodes whole of the flag vector (once all of IH has been found).

Aside: McMullen's conditions and homology

In 1981? McMullen introduced g and conjectured that certain conditions

- linear equations on the f vector (Poincaré duality)
- non-negativity of the g vector (hard Lefshetz)
- pseudopower growth conditions $g_{i+1} \leq g_i^{\langle i \rangle}$ (ring structure)

were necessary and sufficent for f to be the face vector of a convex polytope. His basis was combinatorial intution, not knowledge of homology and algebraic geometry.

In 198? Stanley proved the necessity of McMullen's conditions, provided $H(X)$ satisfied hard Lefschetz, and obtained an argument that showed this.

In 198? an ingenious construction of Billera and Lee proved the sufficiency of McMullen's conditions.

In 1993 McMullen, without using algebraic geometry, proved hard Lefshetz for simple polytopes.

The face lattice and the flag vector

Let X be a convex polytope. It has faces. The intersection of two faces is also a face. (Here we count the empty set as face.) One face might contain another. Hence the polytope X has a face lattice.

Note: Swapping $\subset$ and $\supset$ will swap vertices and facets. It corresponding to the *polarization* of X (about a point in the interior of X).

Definition (Bayer and Billera (1985))

Let X be a d-polytope and I a subset of $\{0, \ldots, d\}$. Each component f_I of the *flag vector* of f counts how many chains

$$\delta = \{\delta_1, \ldots, \delta_r\}, \qquad \delta_i \subset \delta_{i+1}$$

there are in the face lattice of X, for which the set of dimensions of δ is I.

Definition

Let $\mathcal{F}_d$ denote the vector space spanned by all flag vectors of d-dimensional convex polytopes.

The operators I, C and D

Theorem (Bayer & Billera (1985), Generalised Dehn-Sommerville)

The vector space $\mathcal{F}_d$ spanned by the flag vectors of d-polytopes has dimension F_{d+1}, the $(d+1)$st Fibonacci number. It has as basis the flag vectors formed by words in C and IC (where I and C are as below).

Cylinder: For X a convex polytope, the *cylinder* IX on X is the Cartesian product $[0,1] \times X$ of X with an interval.

Cone: For X a convex polytope, the *cone* CX with base X is the join of X with a point not lying in the plane of X.

Theorem (F. (199?), The IC equation)

As operators on flag vectors, $I\,(IC - CC) = (IC - CC)\,I$.

Corollary (The CD basis)

Write $D = IC - CC$. Any linear function on the space $\mathcal{F}_d$ is determined by its value on the words in C and D of degree d.

First-order intersection homology

First-order intersection homology is already known (1978?–1985). Since 1985 I've sought the definition of higher-order intersection homology.

We want to extend the McMullen characterization of f-vectors to general polytopes. For each simple polytopes we have a palindromic h-vector. First-order intersection homology is its extension to general polytopes.

Question

Given say $h(X) = [a, b, c, b, a]$, what are $h(CX)$ and $h(IX)$? We would like $h(CX)$ and $h(IX)$ to be palindromic and unimodal.

If the rules are consistent with the IC equation, then we have defined h on all convex polytopes. The answer is on the next slide.

From the formula for (first-order) intersection homology we can deduce its topological definition. So answering the question amounts to 'discovering intersection homology'.

Why the CD basis?

We want to find a g that encodes the whole of $\mathcal{F}$. But first, let's answer the question from the previous slide (for first-order h and g).

Q: Given say $h(X) = [a, b, c, b, a]$, what are $h(CX)$ and $h(IX)$?

A: $h(CX) = [a, b, c, c, b, a]$ and $h(IX) = [a, a + b, b + c, c + b, b + a, a]$.

Recall $D = IC - CC$. It is easily seen that $h(DX) = [0, a, b, c, b, a, 0]$ and hence that $h(CDX) = h(DCX)$ (for first-order h only).

Proposition (CD formula for first-order g_i)

Let w be a word in C and D, here thought of as a linear combination of polytope flag vectors. The rule for g_i is $g_i(w) = 1$ if w contains exactly i occurences of D, and zero otherwise.

Recall that $\tilde{g}_j$ are the co-ordinate functions for the CD basis of $\mathcal{F}$, and that $\tilde{g}_J(X)$ is defined to be $\sum_{j\in J} \tilde{g}_j(X)$. We've just show that the first-order g_i are equal to $\tilde{g}_J$, for suitable J.

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- Already have CD basis $\tilde{e}_i$ for $\mathcal{F}$. Write $[X] = \sum_{i\in\mathcal{W}} \tilde{g}_i(X)\tilde{e}_i$.
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- ... and this is how to find the $P^i(X)$, and so prove $g_i(X) \geq 0$.
- **Problem:** Find, or guess, the subsets $J(i)$ of $\mathcal{W}$. (Eg M_5 for $d = 5$.)

X is a convex polytope.

$\mathcal{W}$ is Fibonacci words (in C and D).

We already know $\tilde{g}_j(X)$, for all $j \in \mathcal{W}$.

We define $\tilde{g}_J(X) = \sum_{j \in J} \tilde{g}_j(X)$ for $J \subseteq \mathcal{W}$.

We want $g_i(X) = \tilde{g}_J(X)$ for some $J = J(i)$.

We also want $g_i(X) \geq 0$.

If $\tilde{g}_J(X) < 0$ (for some X) then we can't use J.

Eliminate the impossible (dim $= 5$)

We're looking for a nice basis of the space $\mathcal{F}_5$ for 5-polytopes. $\mathcal{F}_5$ has dimension 8. So there are $256 = 2^8$ possible subsets J of the words in CD of degree 5. Each such J gives a linear function $\tilde{g}_J = \sum_{j \in J} \tilde{g}_j$ on $\mathcal{F}_5$.

Question

For which J is $\tilde{g}_J(X) \geq 0$ for all convex polytopes (of dimension 5)?

Definition

If $\tilde{g}_J(X) \geq 0$ for all relevant convex polytopes X, we will say that J is *effective*. (We look only at zero-one polytopes.)

Every time we have $\tilde{g}_J(X) < 0$, for X a convex polytope, we know that J is not effective. As there are $256 = 2^8$ possibilities for J, we need at most 256 values of X to eliminate all that is impossible. Actually, we need rather less. The next 10 or so pages describe what happens.

Proposition 1: $DCCC \implies CDCC$

By $DCCC \implies CDCC$ we mean that if J an effective set of words contains $DCCC$ then it also contains $CDCC$.

Proof: There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$
$$DCCC = -5^*, \quad CDCC = 10^*, \quad CCDC = 0, \quad CCCD = 0$$
$$DDC = 0, \quad DCD = 0, \quad CDD = 0$$

If $DCCC$ is in J then -5 appears in the sum for $\tilde{g}_J(X)$. As J is effective, this forces $CDCC$ to appear in the sum. Thus, $DCCC \implies CDCC$.

Proposition 2: $CDCC \implies CCDC$

By $CDCC \implies CCDC$ we mean that if J is an effective set of words and J contains $CDCC$ then it also contains $CCDC$.

Proof: There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$

$$DCCC = -5, \quad CDCC = -5^*, \quad CCDC = 20^*, \quad CCCD = 0$$

$$DDC = 3, \quad DCD = 0, \quad CDD = 0$$

If $CDCC$ is in J then -5 appears in the sum for $\tilde{g}_J(X)$. As J is effective, this forces $CCDC$ to appear in the sum.

Note: Although X also shows $DCCC \implies CCDC$ this statement, given the previous Proposition, is weaker than and implied by this proposition.

Proposition 3: $CCDC \implies CCCD$

By $CCDC \implies CCCD$ we mean that if J is an effective set of words and J contains $CCDC$ then it also contains $CCCD$.

Proof: There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$
$$DCCC = -1, \quad CDCC = 5, \quad CCDC = -10^*, \quad CCCD = 10$$
$$DDC = 0, \quad DCD = 0, \quad CDD = 0$$

If $CCDC$ is in J then -10 appears in the sum for $\tilde{g}_J(X)$. As J is effective, this forces $CCCD$ to appear in the sum.

Note: We have now proved
$DCCC \implies CDCC \implies CCDC \implies CCCD$.

Proposition 4: $DDC \implies (CDD \text{ or } CDCC)$

By this we we mean that if DDC appears in an effective J then at least one of CDD or $CDCC$ must also appear in J.

Proof: There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$
$$DCCC = 10^*, \quad CDCC = 30^*, \quad CCDC = 0, \quad CCCD = 0$$
$$DDC = -5^*, \quad DCD = 0, \quad CDD = 20^*$$

If DDC is in J then -5 appears in the sum for $\tilde{g}_J(X)$. As J is effective, at least one of $DCCC$, $CDCC$ or CDD must also appear in J. Such an appearance is enough to make J effective.

As $DCCC \implies CDCC$ this reduces the choice to CDD or $CDCC$.

Proposition 5a: $DDC \implies (DCD \text{ or } CCCD)$

Proposition 5b: $DDC \implies (DCD \text{ or } (CCDC \implies DCCC))$

Proof: There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$

$$DCCC = 3, \quad CDCC = 0, \quad CCDC = -10, \quad CCCD = 15$$

$$DDC = -7^*, \quad DCD = 15, \quad CDD = 0$$

If DDC is in J then -7 appears in the sum for $\tilde{g}_J(X)$. Adding DCD to J is sure to make it effective. If we don't add it then we must add $CCCD$, proving part (a). On the same assumption, if we add $CCDC$ we must also add $DCCC$, proving part (b).

Note: Although this example shows $CCDC$ implies ($CCCD$ or DCD), this is not new. It follows from the stronger Proposition 3, $CCDC \implies CCCD$.

Proposition 6: $CCDC \implies (DCD \text{ or } CDCC)$

By this we we mean that if $CCDC$ appears in an effective J then at least one of DCD or $CDCC$ must also appear in J.

Proof: There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$
$$DCCC = -2, \quad CDCC = 10, \quad CCDC = -16^*, \quad CCCD = 14$$
$$DDC = -3, \quad DCD = 6, \quad CDD = 0$$

If $CCDC$ is in J then the -16 appears in the sum for $\tilde{g}_J(X)$. Now take into account

$$DCCC \implies CDCC \implies CCDC \implies CCCD\,.$$

If we have $CCDC$ we must also have $CCCD$, which reduces the negativity from -16 to -2. Our choice now is between DCD or $CDCC$.

Proposition 7: $CDCC \implies (CDD \text{ or } DCCC)$

By this we we mean that if $CDCC$ appears in an effective J then at least one of CDD or $DCCC$ must also appear in J.

Proof: There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$
$$DCCC = 20, \quad CDCC = -66^*, \quad CCDC = 56, \quad CCCD = 8$$
$$DDC = -5, \quad DCD = 0, \quad CDD = 20$$

If $CDCC$ is in J then the -66 appears in the sum for $\tilde{g}_J(X)$. Now take into account

$$DCCC \implies CDCC \implies CCDC \implies CCCD\,.$$

This reduces the negativity to $-66 + 56 + 8 = -2$, which is still negative. So we have to add $DCCC$ or CDD to J.

Note: The polytope X is very special and hard to find.

Propositions 1 to 8

It helps to have a summary here, and the statement of our final proposition.

Proposition 1: $DCCC \implies CDCC$

Proposition 2: $CDCC \implies CCDC$

Proposition 3: $CCDC \implies CCCD$

Proposition 4: $DDC \implies (CDD \text{ or } CDCC)$

Proposition 5a: $DDC \implies (DCD \text{ or } CCCD)$

Proposition 5b: $DDC \implies (DCD \text{ or } (CCDC \implies DCCC))$

Proposition 6: $CCDC \implies (DCD \text{ or } CDCC)$

Proposition 7: $CDCC \implies (CDD \text{ or } DCCC)$

Proposition 8: $(DDC \wedge \neg DCD) \implies (DCCC \implies CDD)$

Here we use logical $\wedge$ (and) and $\neg$ (not).

Proposition 8: $(DDC \wedge \neg DCD) \implies (DCCC \implies CDD)$

Assume that an effective J contains DDC but not DCD. Then

- Prop 4 gives (CDD or $CDCC$).
- Prop 5a gives $CCCD$.
- Prop 5b gives $CCDC \implies DCCC$.
- Prop 6 gives $CCDC \implies CDCC$ (which is weaker than 5b).
- Prop 7 remains $CDCC \implies (CDD$ or $DCCC)$.

There is a convex polytope X with $\tilde{g}$ coefficients:

$$CCCCC = 1$$
$$DCCC = -3, \quad CDCC = 17, \quad CCDC = -26, \quad CCCD = 20$$
$$DDC = -10^*, \quad DCD = 18^\dagger, \quad CDD = 4$$

Now look at the sums for the second row of $\tilde{g}$ coefficients. What's new is that $DCCC$ implies CDD (because $-3 + 17 - 26 + 20 - 10 = -2$).

Aside: Convex Boolean functions [F, 2012]

A Boolean function f is a function whose arguments and value are Booleans. Given integers $c_1, \ldots, c_n$, write $c_J = \sum_{j \in J} c_j$, for each subset J of $N = \{1, \ldots, n\}$. The rule $f_c(J) = (c_J \geq 0)$ gives a *convex* Boolean function. Being effective for a test polytope is an example.

Definition (Convex Boolean function)

Suppose for example J, K, L, M are subsets of N, and that $f(K) = f(L) = f(M) =$ True, and that J is a convex combination of K, L, M. If, whenever such a situation holds we have $f(J) =$ True , we will say that f is a *convex* Boolean function.

(If $K \cap L = \emptyset$ then $K \cup L$ is a *convex combination* of K and L. Similarly, $\{1, 2, 3\}$ is a *convex combination* of $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$.)

It seems that nothing is known about convex Boolean functions. (Ekin, Hammer and Kogan (2000) use the same name for a different concept.)

Effective J in dimension 5

Any J effective on the above test polytopes is also effective on all zero-one polytopes. All such J are listed below (omitting trivial consequences):

- $g_0 : CCCCC$
- $g_1 : DCCC, CDCC, CCDC, CCCD$
- $g_2 : DDC, DCD, CDD$
- $g_{1211} : CDCC, CCDC, CCCD, CDD^*$ ($*$: the special entry in M_5)
- $g_{1121} : CCDC, CCCD, DCD$
- $g_{1112} : CCCD$
- $g_{122} : CDD$
- $g_{212} : DCD$ (there are 8 J listed here)

and the J that express the 5 inequalities

- $g_{122} - g_{221} \leq \min(g_{2111}, g_{1211})$
- $g_{212} - g_{221} \leq \min(g_{2111}, g_{1121}, g_{1112})$

Why *CDD* in addition to *CDCC*, *CCDC*, *CCCD*?

Here's a conjectural explanation of $\{CDCC, CCDC, CCCD\}$ why requires *CDD* to be effective. It is related to **locality** and interaction.

A word about (co)homology. It's about differential forms (integrands), regions of integration, Stoke's theorem and periods. Think of the two types of circle on an inner tube (or bagel).

Non-simple polytopes have local homology. The terms *CDCC*, *CCDC*, *CCCD* produce 1-cycles **local to a movable point**, but this homology space is not pure. It seems that adding *CDD*, which is 2-cycle local to a point, restores purity. But 1-cycles and 2-cycles cannot, as such, interact.

Hard Lefshetz allows us to raise dimension. Apply it to a local 1-cycle and we obtain a 2-*cycle whose image under* ω *is localised to a point*. This, an object that is **not local to a point**, can now interact with *CDD*-cycles.

This requires ω, which roughly speaking is a system of coordinates on X. So, it seems, there is pure homology that is not absolute. A puzzle.

Higher-order subset polytopes

Recall that the Fibonacci polytope vertices are **special vertices** of a cube, and that we are looking for **special subsets** of the vertices of the Fibonacci polytope. Perhaps the special subsets are also polytope vertices.

Let D be a finite set D. Recall that the **elements** of D are the vertices of the **simplex** on D, and that the **subsets** of D are the vertices of the **cube** on D. Can we continue this sequence? What about **subsets of subsets**?

Definition (Higher-order subset polytope (incomplete))

The zero order subset polytope $\mathrm{Subset}^0(D)$ on a finite set D is the simplex on D. The order-n subset polytope $\mathrm{Subset}^n(D)$ is the convex polytope, whose vertices are the subsets of the the vertices of $\mathrm{Subset}^{n-1}(D)$, constructed by a procedure we have not yet defined.

Question

Are the special subsets we are looking for the vertices of a Fibonacci polytope (in Subset^2*(Fibonacci words))?*

Does everything have a polytope?

In mathematics, every is (can be represented by) a higher-order subset (of the empty set). In Zermelo-Fraenkel set theory we have

$$3 = \{0, 1, 2\} = \{\, \{\}, \quad \{\{\}\}, \quad \{\{\}, \{\{\}\}\}\, \}\,.$$

Around 1850 mathematics started to define a function as a special subset of the Cartesian product of two sets (which relies on ordered pairs).

Here are three 'foundational' definitions of an ordered pair

- $(a, b) = \{\{\{a\}, \{\}\}, \{\{b\}\}\}$, due to Norbert Wiener, 1914.
- $(a, b) = \{\{a, 1\}, \{b, 2\}\}$, due to Felix Hausdorff, 1914.
- $(a, b) = \{\{a\}, \{a, b\}\}$, due to Kazimierz Kuratowski, 1921.

Question

Does the polytope of an object (once we have defined higher-order polytopes) depend on the 'foundational data model' used for integers, ordered pairs etc?