Candidate Betti numbers for the linear homology of convex polytopes

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Abstract

The middle perversity intersection homology (mpih) Betti numbers of the toric variety associated with a convex polytope are *linear functions* of the *flag vector* of the convex polytope.

In this talk I define *similar linear functions*, which I hope are the Betti numbers for a not yet defined homology theory. This *linear homology* theory should *exist wherever mpih does*.

Such homology would prove that these candidate Betti numbers are actual Betti numbers, and so *non-negative on all convex polytopes* (being the dimension of a vector space).

Overview

This talk has three sections

- §1 Preliminaries (8 slides)
- §2 Definition of candidate Betti numbers (9 slides)
- §3 Flag vectors and other loose ends (8 slides)

Topics for further work include

- Test polytopes and their construction
- The combinatorics of candidate Betti numbers
- Construction of the linear homology of convex polytopes
- Classical topology and linear homology

After all this, someone might be ready to prove that the candidate Betti numbers are actual Betti numbers (and so non-negative).

§1 Preliminaries

In this section we set the stage for the definition of the candidate Betti numbers. The slides in this section are

- Top-down and just-in-time
- The title explained
- Simple polytopes
- Problem: complete the square
- The *CD* basis for $\mathcal{F}_q$
- Middle perversity intersection homology (mpih)
- Zero-one linear functions and candidate Betti numbers
- Fibonacci indices

The next section is *§2 Definition of candidate Betti numbers*.

Top-down and just-in-time

I'm taking a 'top-down' and 'just-in-time' approach, explaining the key ideas first and leaving the details to later.

Don't worry that not everything is explained right now. I want to show you the forest, not the trees.

We use **bold text** for a term that is being defined now, or will be or ought to be defined later (or which you might have to look up afterwards).

But you ought to know that a **convex polytope** X is the image of a simplex under a linear map (or the **convex hull** of a finite set of points).

And that X is **simple** if it has exactly d edges at each vertex (where d is the dimension of the polytope). Non-simple vertices are **singular**.

The **cone on a square**, as in an Egyptian pyramid, is singular at the **apex** because there it has 4 edges.

The title explained

Homology and Betti numbers

- By **homology** we mean a geometric process H that constructs, from X, finite dimensional vector spaces $H_i(X)$.
- Every $H_i(X)$ has a **dimension**, which is a non-negative integer.
- The dimension of $H_i(X)$ is called the i-th **Betti number** $h_i(X)$ of X.

Linear homology and candidate Betti numbers

- We write 'd-polytope' for 'd-dimensional convex polytope'.
- Every d-polytope X has a **flag vector** $f(X)$. These flag vectors span a vector space $\mathcal{F}_d$.
- Homology is **linear** if $h(X)$ is a linear function of $f(X)$.
- We say that a function h on $\mathcal{F}_d$ provides **candidate Betti numbers** if we hope to solve $h_i(X) = \dim H_i(X)$ for H.

Simple polytopes

Let X be a **simple** d-polytope. At each vertex, X has exactly d edges.

Let α be a generic linear height function defined on X. The **index** $\operatorname{ind}(v) = \operatorname{ind}_\alpha(v)$ at a vertex v is X is the number of downward edges at v.

Let's count each face of X at its highest vertex. The faces counted at v correspond to subsets of the downward edges at v (because X is simple at v). Let $h_i(X)$ be the number of index i vertices on X. This shows:

Proposition We have $f(X) = Ch(X)$, where C is an upper-triangular matrix of binomial coefficients, and $f(X)$ is the face vector.

Because C is invertible we can compute $h(X)$ from $f(X)$, and so $h(X)$ does not depend on the choice of α. Replace α by $-\alpha$ to swap up and down, and hence reverse $h(X)$. This proves:

Theorem The vector $h(X)$ is palindromic.

This is McMullen's counting argument for the **Dehn-Sommerville equations**. It is also Morse theory for the associated toric variety.

Problem: Complete the square

$$\begin{array}{ccccc} [\text{simple}] & \rightarrow & [\text{gDS}] & = & [\text{start}] \\ \downarrow & & \downarrow & & \\ [\text{mpih}] & \dashrightarrow & [\text{general}] & = & [\text{goal}] \end{array}$$

[simple] Ring generated by facets + Poincaré duality (Pd) + hard Lefschetz + combinatorics + construction $\Longrightarrow$ McMullen's conjectured necessary and sufficient conditions (nasc) on $f(X)$, for simple polytopes.

[general] Find and prove nasc on $f(X)$, for general polytopes.

[gDS] Generalized Dehn-Sommerville (weak Pd) for all of $f(X)$.

[mpih] Pd and hard Lefschetz for small part of $f(X)$, but no ring structure.

Since 1985, I've worked on this. In 2013 I found candidate Betti numbers.

This talk starts at [gDS], and goes parallel to [simple] → [mpih].

There's lots left to do.

The CD basis for $\mathcal{F}_d$

We seek a special linear function on $\mathcal{F}_d$. A nice basis helps.

There are geometric linear operators C and I of degree 1. They map $\mathcal{F}_d$ to $\mathcal{F}_{d+1}$. Further, $\{f(\mathrm{pt})\}$ is basis for $\mathcal{F}_0$, and on it C and I are equal. (C is the **cone** or pyramid operator, and I is the **cylinder** or prism operator. They act on polytopes and descend to $\mathcal{F}$.)

Theorem Generalised Dehn-Sommerville (Bayer and Billera, 1985)
The operators C and IC generate a basis for $\mathcal{F}_d$, which hence has dimension the $(d+1)$-st Fibonacci number F_{d+1}.

Theorem IC-equation (F, 1995)
The operators I and $IC - CC$ commute.

Definition Write $D = IC - CC$. The words in C and D of degree d, applied to a point, generate the CD-**basis** for $\mathcal{F}_d$.

Middle perversity intersection homology (mpih)

Every d-polytope X with rational vertices determines a projective algebraic variety $\mathbb{P}_X$, a **toric variety**. All projective varieties have mpih.

Several people have used the **decomposition theorem** to prove:

Theorem (Bernstein, Khovanskii, MacPherson, ..., circa 1982)
The mpih Betti numbers $h_i(X)$ of $\mathbb{P}_X$ are explicit linear functions of $f(X)$.

They gave a complicated (and useful) recursive formula for h_i. But we can simplify things, using the CD-basis.

Following **hard Lefschetz** we write $g_i = h_i - h_{i-1}$ for $2i \leq d$.

Corollary Let w be a word in C and D. Let $X = w(\mathrm{pt})$ be the associated formal sum of toric varieties. Then $g_i(X) = 1$ if D occurs exactly i times in w, and is zero otherwise. (And h_i similarly for D at most i times.)

In other words, g_i is zero-one in the CD-basis (as so is h_i).

Zero-one linear functions and candidate Betti numbers

A linear function ℓ is determined by its value on a basis W.

Definition The linear function ℓ is a **zero-one** (with respect to the basis W) if on each vector $w \in W$ it is either zero or one.

Note Such determines the subset $\{\ell(w) = 1\}$ of W, and vice versa.

Example The mphi g_i are zero-one linear functions (for the CD basis). The set of words with exactly i occurences of D gives g_i.

For geometric reasons, arising for the geometry of mpih cycles on toric varieties, particularly those constructed using C and I:

We assume that the candidate Betti numbers are zero-one linear functions for the CD-basis.

Thus, we seek special subsets of the CD-basis. To make this easier, we use Fibonacci indices (see next slide) instead of words.

Fibonacci indices

Let w be a word in C and D. The **length** of w is the number of letters. The **offset** is the number of D's. (The **degree** is length + offset).

It is the operator CD that creates new **singularities**. The **depth** is the number of CD's. The depth is important.

If the depth is zero then $w = D^i C^j$ (and we say w is **simple**). We associate to such a w the symbol $\left[{a \atop b} \right]$ where $a = i + j$ is the length, and $b = i$ is the offset (and $a + b$ is the degree).

Lemma Each w is uniquely a minimal product of simple words.

Definition Write w as a minimal product of k simple words w_i. Each of these factors has a length a_i and an offset b_i. The symbol

$$\begin{bmatrix} a_1 \; a_2 & & a_k \\ b_1 \; b_2 & \cdots & b_k \end{bmatrix}$$

is the **Fibonacci index** (or **fib-ind**) of w. To be a fib-ind it must arise in this way. The depth is $k - 1$ (unless w is empty).

§2 Candidate Betti numbers

In this section we define the candidate Betti numbers. They are more complicated than mpih. The slides in this section are

- About candidate Betti numbers
- The $d = 5$ candidate Betti numbers (1 / 2)
- The $d = 5$ candidate Betti numbers (2 / 2)
- Not homogeneous — the offset varies
- The split partial order $>_{\text{split}}$
- The left partial order $>_{\text{left}}$
- Not transitive — examples
- The offset partial order $>_{\text{offset}}$
- **The candidate Betti numbers**

The next and final section is *§3 Flag vectors and other loose ends*.

About candidate Betti numbers

Summary Let I_d denote the set of all degree-d Fibonacci indices. Each subset s of I_d gives rise to a linear function ℓ_s on the vector space $\mathcal{F}_d$. We expect the candidate Betti numbers to have this form.

Remark Suppose X is a d-polytope and s is a subset of I_d. *If $\ell_s(X) < 0$ then ℓ_s cannot be a Betti number* (as genuine Betti numbers are non-negative). This allows us to eliminate many candidates.

We might also expect the candidate Betti numbers to provide a basis for $\mathcal{F}_d$. Thus, we seek a function cand that associates to each Fibonacci index $i \in I_d$ a subset $s = \text{cand}(i)$ of I_d. *Such a function could be a candidate Betti number* (if not eliminated by $\ell_s(X) < 0$).

For $d = 5$ a long calculation produces the candidate Betti numbers, presented on the next two slides. After that, we extrapolate (or guess).

It's not practical to do the same calculation in $d = 6$, it's too big.

The $d = 5$ candidate Betti numbers (1 / 2)

We want $\dim \mathcal{F}_5 = 8$ candidate subsets. Three of them (below) come from mpih. They arise by counting D's.

A long calculation (see slides 14 and 24) produces the remaining five, which are on the next slide. We're looking for a pattern.

fib-ind i	subset cand(i)	associated CD-words
$\begin{bmatrix}5\\0\end{bmatrix}$	$\begin{bmatrix}5\\0\end{bmatrix}$	$CCCCC$
$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}1&3\\0&1\end{bmatrix}, \begin{bmatrix}2&2\\0&1\end{bmatrix}, \begin{bmatrix}3&1\\0&1\end{bmatrix}$	$DCCC, CDCC, CCDC, CCCD$
$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}1&2\\1&1\end{bmatrix}, \begin{bmatrix}1&2\\0&2\end{bmatrix}$	DDC, DCD, CDD

The $d = 5$ candidate Betti numbers (2 / 2)

fib-ind i	subset cand(i)	associated CD-words
$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	$CDCC, CCDC, CCCD, \ CDD$
$\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	$CCDC, CCCD, \ DCD$
$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$	$CCCD$
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	CDD
$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	DCD

Not homogeneous — the offset varies

Our task is to extrapolate the pattern for $d = 5$. We can also use the mpih formula for all d.

Mostly, in the $d = 5$ candidate, both i and the elements of cand(i) have the same offset (sum of numbers on bottom row).

But there are two exceptions:

$$\text{cand}\left(\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}\right) \text{ contains } \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{i.e. } CCDC, DCD$$

$$\text{cand}\left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\right) \text{ contains } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \qquad \text{i.e. } CDCC, CDD$$

Finding (and interpreting) the non-homogeneous part of the pattern is harder than the rest. The next two slides give the homogeneous part.

The split partial order $>_{\text{split}}$

Let i be the fib-ind $\begin{bmatrix} a \\ b \end{bmatrix}$. It has depth one. This corresponds to an mpih Betti number, namely $g_b = \text{cand}(i)$ on $(a+b)$-dimensional polytopes.

The fib-inds in $\text{cand}(i)$ with depth ≥ 1 are obtained by splitting i.

Definition The **split partial order** is generated by

$$\begin{bmatrix} \ldots & a+b & \ldots \\ & c+d & \end{bmatrix} >_{\text{split}} \begin{bmatrix} \ldots & a \; b & \ldots \\ & c \; d & \end{bmatrix}$$

provided both sides are Fibonacci indices.

We read this as saying the LHS can be **split** to give the RHS.

Lemma The mpih Betti numbers are produced by applying the split partial order to indices of the form $\begin{bmatrix} a \\ b \end{bmatrix}$.

The left partial order $>_{\text{left}}$

The offset = 1 indices in

$$\text{cand}\left(\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\right) = \left\{\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}, \begin{bmatrix}2 & 2\\0 & 1\end{bmatrix}, \begin{bmatrix}3 & 1\\0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\right\}$$

are obtained by moving to the left some of the amount in the top row.

Definition The **left partial order** is generated by

$$\begin{bmatrix}\dots & a & \dots & b' & \dots\\ & c & & d & \end{bmatrix} >_{\text{left}} \begin{bmatrix}\dots & a' & \dots & b & \dots\\ & c & & d & \end{bmatrix}$$

provided (this is the leftward motion) $b' - b = a' - a > 0$, and also that both sides are Fibonacci indices.

Proposition For $d = 5$, the following are equivalent:

- We have $j \in \text{cand}(i)$, and j has the same offset as i.
- We can solve $i \geq_{\text{left}} x \geq_{\text{split}} j$ for $x \in I_d$.

Not transitive — examples

We have, using the CD notation,

$$\begin{aligned} \text{cand}(CDCC) &\text{ contains } CCDC \\ \text{cand}(CCCD) &\text{ contains } DCD \\ \text{cand}(CDCC) &\text{ does not contain } DCD \end{aligned}$$

and also

$$\begin{aligned} \text{cand}(DCCC) &\text{ contains } CDCC \\ \text{cand}(CDCC) &\text{ contains } CDD \\ \text{cand}(DCCC) &\text{ does not contain } CDD \end{aligned}$$

and thus belonging to a candidate subset is not a transitive relation.

We have to be careful when changing the offset.

Aside How does this relate to intersection homology? See slides 27, 28.

The offset partial order $>_{\text{offset}}$

The essence of the $>_{\text{offset}}$ partial order is that we can increase offset, perhaps moving left as we do. Two C's combine to give a D. But there is a condition, namely not both of the C's come from the first column.

Definition The **offset partial order** is generated by

$$\begin{bmatrix} \dots & a & \dots & b' & \dots \\ & c & & d & \end{bmatrix} >_{\text{offset}} \begin{bmatrix} \dots & a & \dots & b & \dots \\ & c' & & d & \end{bmatrix}$$

$$\begin{bmatrix} \dots & a & b' & \dots \\ & c & d & \end{bmatrix} >_{\text{offset}} \begin{bmatrix} \dots & a & b & \dots \\ & c & d' & \end{bmatrix}$$

provided (this is the increase in offset) $b' - b = c' - c = d' - d > 0$, and also that both sides are Fibonacci indices.

Note These rules give $CCDC >_{\text{offset}} DCD$, and $CDCC >_{\text{offset}} CDD$.

Note If i is a simple (mpih) then $i >_{\text{offset}} j$ has no solutions.

Note The relation $CCCD >_{\text{offset}} j$ has no solutions.

The candidate Betti numbers

We can now put everything together, but must take care with the order

Recall that we have Fibonacci indices I_d, and on the Fibonacci indices we have the split, left and offset partial orders. Recall also that each subset s of the Fibonacci indices produces a linear function ℓ_s on $\mathcal{F}_d$.

Definition For each i in I_d we define $s = \text{cand}(i)$ to be all j in I_d such that the equation

$$i \quad \geq_{\text{offset}} \quad x \quad \geq_{\text{left}} \quad y \quad \geq_{\text{split}} \quad j$$

can be solved, for x and y in I_d.

In other words, from i we can first increase offset some, then move left some, and then split. But the changes must be done in that order.

Definition We call this ℓ_s the ***i*-th candidate Betti number** g_i.

Lemma For $d = 5$, this definition agrees with the explicit candidate Betti numbers given earlier.

We've reached our goal. Please relax.

§3 Flag vectors and other loose ends

In this section we define flag vectors and deal with various loose ends.

- The long $d = 5$ calculation
- A very special 5-polytope
- Local 1-cycles
- The first offset rule
- The mysterious second offset rule
- The concrete flag vector
- The abstract flag vector
- Bonus: Quadratic homology

This is the final section of this talk.

The long $d = 5$ calculation

From a **test set** of $\mathcal{T}$ of d-polytopes, we can get candidate Betti numbers.

1. Let $\mathcal{S}$ be all $s \subseteq I_d$ such that $\ell_s(X) \geq 0$ for all X in $\mathcal{T}$.
2. Say $s \in \mathcal{S}$ is **extremal** if it is not implied by the other members of S.
3. A **candidate Betti number** is an extremal $s \in S$ whose ℓ_s *either* is a mpih Betti number *or* vanishes on simple polytopes.

Proposition For each d there is a finite set $\mathcal{T}_d$ of test polytopes such that the resulting candidate Betti numbers are never negative on d-polytopes.

Polytopes whose facets (resp. vertices) are chosen are random are usually simple (resp. simplicial). Instead of random choice we use brute force.

Definition A polytope whose vertices are a subset of the vertices of a cube is a **zero-one** polytope.

For $d = 5$, I used all zero-one 5-polytopes, along with their polars. This calculation takes about 15 GHz-days [F, 2010].

A very special 5-polytope

The $d = 5$ candidates come from zero-one test polytopes.

Proposition There is a zero-one 5-polytope X with CD vector:

$$CCCCC = 1$$
$$DCCC = 20, \quad CDCC = -66^*, \quad CCDC = 56, \quad CCCD = 8$$
$$DDC = -5, \quad DCD = 0, \quad CDD = 20$$

Corollary Suppose $\ell_s(X) \geq 0$ and $CDCC \in s$ and $DCCC \notin s$ (and so s is not mphi). Then $CDD \in s$. (This gives part of cand($CDCC$).)

What is X? The 5 cube has $32 = 2^5$ vertices, so the 32-bit number 0110 1111 1111 0110 1111 1001 1001 1111 gives a **cube vertex subset**.

The polytope X is the **polar** of the convex hull of this cube vertex subset. Up to symmetry, it's the only zero-one way to get the corollary.

We found the hard to find. Perhaps we have enough test polytopes.

Local 1-cycles

Local 1-cycles and their homology can help us understand the offset rules.

Definition Let δ be a facet of X. A **spike** of δ is an affine linear function on X that vanishes on δ.

Definition A **1-cycle on** X associates a spike of δ to each facet δ of X in such a way that the sum of the linear functions is constant.

Let X be the cone on a square. We can construct a 1-cycle on X using only the facets that pass through the apex. (Four facets, three conditions.)

Definition A **local 1-cycle** on X is a proper face δ of X together with a 1-cycle that spikes only the facets through δ.

A local 1-cycle along an edge induces local 1-cycles at each end of the edge (by 'extending the association by zero').

Definition The **local-global 1-homology** on X consists of formal sums of vertex-based local homology, modulo equivalence along edges.

The first offset rule

This slide and the next start to relate the combinatorics to the geometry.

The first offset rule is

$$\begin{bmatrix} \cdots & a & \cdots & b' & \cdots \\ & c & & d & \end{bmatrix} >_{\text{offset}} \begin{bmatrix} \cdots & a & \cdots & b & \cdots \\ & c' & & d & \end{bmatrix}$$

and it gives $CCDC >_{\text{offset}} DCD$.

In the example above $CCDC$ corresponds to a local 1-cycle (the trailing DC) present along a 1-dimensional cycle (the leading CC).

Also in this example, DCD corresponds to a local 1-cycle (the trailing D) present along a 1-dimensional cycle (the leading DC).

Both are the same sort of thing, so perhaps count them together.

In the language of geometry, in both cases we have something like a 1-cycle with coefficients in local 1-cycles.

This also explains $CCCD \not>_{\text{offset}} DCD$ (because $CCCD$ is a 2-cycle).

The mysterious second offset rule

The second offset rule is

$$\left[\cdots \begin{matrix} a & b' \\ c & d \end{matrix} \cdots\right] \succ_{\text{offset}} \left[\cdots \begin{matrix} a & b \\ c & d' \end{matrix} \cdots\right]$$

and it gives $CDCC \succ_{\text{offset}} CDD$. Geometrically, why the CDD?

The very special polytope X has $CDCC = -66$, $CCDC = 56$ and $CCCD = 8$. It also has $CDD = 20$. What's the local-global meaning of this?

We have $-66 + 56 \times 2 + 8 \times 3 = 70$ generators, $56 + 8 \times 3 = 80$ relations, and 8 relations among relations (or 2-relations). We have

$$-66 + 56 + 8 = -2 = 70 - 80 + 8$$

and so we need some more 2-relations (or somehow generators).

Suppose there is a 2-relation, local to a vertex, among 1-relations. So far these are not counted. However, CDD counts local 2-cycles.

Conjecture Each local 2-relation on X produces a local 2-cycle.

The concrete flag vector

Let X be a d-polytope. The concrete flag vector counts faces recursively.

Definition A **flag** on X is a chain δ

$$\delta_1 \subsetneq \delta_2 \ldots \subsetneq \delta_k \subsetneq X$$

of faces of S. Its **dimension vector** t is the tuple

$$t = (\dim \delta_1 < \dim \delta_2 \ldots < \dim \delta_k < d)$$

Definition The **concrete flag vector** $f = f_c(X)$ counts the flags on X by their dimension vector t. We write $f_t(X)$ for the number of t-flags on X.

Clearly, the concrete flag vector $f_c(X)$ gives a tuple of 2^d numbers. The equations that exist between these numbers are called the generalised Dehn-Sommerville equations. They were discovered and proved by Bayer and Billera in 1985, and inspired the search for linear homology.

For X simple, the flag vector is a linear function of the face vector.

The abstract flag vector

Let $\mathcal{V}_d$ be vector space of finite formal sums of d-polytopes. Let $\mathcal{F}_d$ be vector space spanned by (concrete) flag vectors. The map $f : \mathcal{V}_d \to \mathcal{F}_d$ is linear. The abstract flag vector describes the kernel $f^{-1}(0)$ of f.

Recall inclusion-exclusion counting $\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$.

Theorem (F, 1995) Let X be a convex polytope divided into three pieces by two hyperplanes in general position. Then

$$f(X) - f(X_1) - f(X_2) + f(X_{12}) = 0$$

where X is $A \cup B$, X_{12} is $A \cap B$ etc. Proof is by inclusion-exclusion.

This is X with two independent changes (X_1 and X_2) applied to it. Hyperplane truncation is not the only sort of independent change.

Theorem? (F, in progress) The relations $[X] - [X_1] - [X_2] + [X_{12}]$, for all independent pairs (X_1, X_2), span the kernel of $f : \mathcal{V}_d \to \mathcal{F}_d$.

Definition The quotient ($\mathcal{V}_d$ mod relations) is the **abstract flag vector**.

Bonus: Quadratic homology

Motivated by definition of Vassiliev knot invariants. Relation to quadratic homology of Baues unknown.

Definition A finite set $S = \{X_i\}$ of changes to X are **independent** if they do not overlap, i.e. any subset s of S defines a composite change X_s of X.

Example On a plane knot diagram, crossing changes are independent.

Definition A function $\ell(X)$ is **linear** if it satisfies the four-term equation

$$\ell(X) - \ell(X_1) - \ell(X_2) + \ell(X_{12}) = 0$$

for every pair (X_1, X_2) of independent changes.

Question For polytopes equivalent to linear function of the flag vector?

Definition A function $q(X)$ is **quadratic** if it satisfies the eight term equation for every triple (X_1, X_2, X_3) of independent changes

Problem Partially complete the cube, by defining $Q(X)$ and so on.

I hope quadratic homology arrives quicker than linear homology.

The End

Trailer Topics for further work include

- Test polytopes and their construction
- The combinatorics of candidate Betti numbers
- Construction of the linear homology of convex polytopes
- Classical topology and linear homology