Finding linear homology

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Abstract

The middle perversity intersection homology (mpih) Betti numbers of the toric variety associated with a convex polytope are *linear functions* of the *flag vector* of the convex polytope.

In this talk I define *similar linear functions*, which I hope are the Betti numbers for a not yet defined homology theory. This *linear homology* theory should *exist wherever mpih does*.

Such homology would prove that these candidate Betti numbers are actual Betti numbers, and so *non-negative on all convex polytopes* (being the dimension of a vector space).

What is linear homology?

What is homology?

Homology is a geometric process H that constructs finite dimensional vector spaces $H_i(X)$, for suitable X and i.

The dimension of $H_i(X)$ is called the i-th **Betti number** $h_i(X)$ of X.

The Betti numbers $h_i(X)$ are, of course, non-negative.

What is linear?

For several types of objects we have a concept of a **linear function**:

- Numbers, and more generally vectors.
- Convex polytopes, via the flag vector (next slide).
- Algebraic varieties over finite fields, via counting points.
- Algebraic varieties, via independent changes (F, in progress).

Homology is **linear** when $h_i(X)$ is a linear function of X (and $h_i(X) \geq 0$).

The (combinatorial) flag vector

Throughout X will be a d-dimensional convex polytope, or d-polytope for short. The (combinatorial) flag vector counts faces recursively.

Definition A **flag** on X is a chain δ

$$\delta_1 \subsetneq \delta_2 \ldots \subsetneq \delta_k \subsetneq X$$

of faces of S. Its **dimension vector** t is the tuple

$$t = (\dim \delta_1 < \dim \delta_2 \ldots < \dim \delta_k < d)$$

Definition The **flag vector** $f = f(X)$ counts the flags on X by their dimension vector t. We write $f_t(X)$ for the number of t-flags.

Clearly, $f(X)$ gives a tuple of 2^d numbers. Linear equations exist between these numbers. They are the generalised Dehn-Sommerville equations (discovered and proved by Bayer and Billera in 1985). This and mpih inspired the search for linear homology.

We write $\mathcal{F}_d$ for the span of $f(X)$ for all d-polytopes X.

The CD basis for $\mathcal{F}_d$

We seek special linear functions on $\mathcal{F}_d$. A nice basis helps.

There are geometric linear operators C and I of degree 1. They map $\mathcal{F}_d$ to $\mathcal{F}_{d+1}$. Further, $\{f(\text{pt})\}$ is basis for $\mathcal{F}_0$, and on it C and I are equal.

(C is the **cone** or pyramid operator, and I is the **cylinder** or prism operator. They act on polytopes and descend to $\mathcal{F}$.)

Theorem Generalised Dehn-Sommerville (Bayer and Billera, 1985)
The operators C and IC generate a basis for $\mathcal{F}_d$, which hence has dimension the $(d+1)$-st Fibonacci number F_{d+1}.

Theorem IC-equation (F, 1995)
The operators I and $IC - CC$ commute on $\mathcal{F}$ (because $IC - CC$ is the difference between multiplying by a square and by a triangle).

Definition Write $D = IC - CC$. The words in C and D of degree d, applied to a point, generate the CD-**basis** for $\mathcal{F}_d$.

Finding mpih (the easy way)

We use mpih as an abbreviation for middle perversity intersection homology (discovered by Goresky and MacPherson, 1977).

It is linear — various authors, based on extension (1982) by Beilinson, Bernstein, Deligne and Gabber of Deligne's 1980 proof of the Weil conjectures (1949).

The mpih h-vector $h(X) = [h_0(X), \ldots, h_d(X)]$ is **palindromic** (Poincaré duality, $h_i = h_{d-i}$) and **unimodal** (hard Lefschetz, single hump).

Suppose $v = [a, b, c, b, a]$ is palindromic and unimodal. Then both of

- $Cv = [a, b, c, c, b, a]$
- $Iv = [a, a+b, b+c, c+b, b+a, a]$

are palindromic and unimodal. Because these C and I satisfy the IC equation (because $Dv = [0, a, b, c, b, a, 0]$) we get a linear function on $\mathcal{F}$, which is the mphi h-vector. (So the formula for h is not so hard.)

Unpack the formula for mpih h_i to obtain the definition (only) of mpih.

Linear homology and the CD basis

For mpih we 'guessed' formulas for C and I. For linear homology we use non-negativity and the CD basis. There's still some guessing.

A linear function ℓ is determined by its value on a basis W (eg CD basis).

Definition The linear function ℓ is a **zero-one** (with respect to the basis W) if on each vector $w \in W$ it is either zero or one.

Note ℓ determines $s = \{\ell(w) = 1\} \subseteq W$. Conversely s determines $\ell_s = \ell$.

Example The mphi h_i are zero-one linear functions (for the CD basis). The set of words with exactly i occurences of D gives $g_i = h_i - h_{i-1}$.

For geometric reasons, arising for the geometry of mpih cycles on toric varieties, particularly those constructed using C and I:

We assume that the linear homology Betti numbers are zero-one for the CD-basis.

Thus, we seek special subsets of the CD-basis. To make this easier, we use Fibonacci indices (a few slides on) instead of words.

The long $d = 5$ calculation

From a **test set** of $\mathcal{T}$ of d-polytopes, we can get candidate Betti numbers.

Let W_d be the CD basis for $\mathcal{F}_d$.

1. Let $\mathcal{S}$ be all $s \subseteq W_d$ such that $\ell_s(X) \geq 0$ for all X in $\mathcal{T}$.
2. Say $s \in \mathcal{S}$ is **extremal** if it is not implied by the other members of S.
3. A **candidate Betti number** is an extremal $s \in S$ (whose ℓ_s *either* is a mpih Betti number *or* vanishes on simple polytopes – we see why later).

Proposition For each d there is a finite set $\mathcal{T}_d$ of test polytopes such that the resulting candidate Betti numbers are never negative on d-polytopes.

Polytopes whose facets (resp. vertices) are chosen are random are usually simple (resp. simplicial). Instead of random choice we use brute force.

Definition A polytope whose vertices are a subset of the vertices of a cube is a **zero-one** polytope.

For $d = 5$, I used all zero-one 5-polytopes, along with their polars. This calculation takes about 15 GHz-days [F, 2010].

A very special 5-polytope

Proposition There is a 5-polytope X with CD basis coefficients:

$$CCCCC : 1$$
$$DCCC : \mathbf{20}, \quad CDCC : -\mathbf{66}, \quad CCDC : \mathbf{56}, \quad CCCD : 8$$
$$DDC : -5, \quad DCD : 0, \quad CDD : \mathbf{20}$$

Remark The dimension of a vector space is non-negative.

Question Which sums of coefficients are non-negative?

Answer If $CDCC$ is used, then so is $CCDC$ and if not $DCCC$ (and so mpih) then CDD. There's also conditions involving DDC.

What is X? The 5 cube has $32 = 2^5$ vertices, so the 32-bit number 0110 1111 1111 0110 1111 1001 1001 1111 gives a **cube vertex subset**.

The polytope X is the **polar** of the convex hull of this cube vertex subset. Up to symmetry, it's the only zero-one way to get the proposition.

Fibonacci indices

Every **Fibonacci index** arises as follows. Here's a word in C and D

$$DCDDCDCCCCDDDDCDDDD$$

(swimming before the eyes) which we split at each occurrence of CD

$$DC \qquad DDC \qquad DCCCC \qquad DDDDC \qquad DDDD$$

and then we count the **length** = a_i and **offset** = b_i of each piece

$$\overbrace{\underbrace{D}_{1}\,C}^{2} \qquad \overbrace{\underbrace{DD}_{2}\,C}^{3} \qquad \overbrace{\underbrace{D}_{1}\,CCCC}^{5} \qquad \overbrace{\underbrace{DDDD}_{4}\,C}^{5} \qquad \overbrace{\underbrace{DDDD}_{4}}^{4}$$

and the symbol

$$\begin{bmatrix} 2\,3\,5\,5\,4 \\ 1\,2\,1\,4\,4 \end{bmatrix} = \begin{bmatrix} a_1 & \ldots & a_5 \\ b_1 & & b_5 \end{bmatrix}$$

is a Fibonacci index of **degree** $\sum(a_i + b_i) = 31$. Except at ends, $a_i > b_i > 0$.

The $d = 5$ candidate Betti numbers (1 / 2)

In $d = 5$ we use zero-one test polytopes to produce candidate linear homology Betti numbers (see slides 8 and 9).

We get $\dim \mathcal{F}_5 = 8$ candidate subsets. Those below come from mpih, and count D's (total offset). The remainder are on the next slide.

fib-ind i	subset $s = \mathrm{cand}(i)$	associated CD-words
$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$	$CCCCC$
$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$	$DCCC, CDCC, CCDC, CCCD$
$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	DDC, DCD, CDD

The $d = 5$ candidate Betti numbers (2 / 2)

fib-ind i	subset $s = \text{cand}(i)$	associated CD-words
$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	$CDCC, CCDC, CCCD, \quad CDD$
$\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	$CCDC, CCCD, \quad DCD$
$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$	$CCCD$
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	CDD
$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	DCD

Not homogeneous — the offset varies

Our task is to extrapolate the pattern for $d = 5$. We can also use the mpih formula for all d.

Mostly, in the $d = 5$ candidate, both i and the elements of $\mathrm{cand}(i)$ have the same offset (sum of numbers on bottom row).

But there are two exceptions:

$$\mathrm{cand}\left(\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}\right) \text{ contains } \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{i.e. } CCDC, DCD$$

$$\mathrm{cand}\left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\right) \text{ contains } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \qquad \text{i.e. } CDCC, CDD$$

Finding (and interpreting) the non-homogeneous part of the pattern is harder than the rest. The next two slides give the homogeneous part.

The split partial order $>_{\text{split}}$

Let i be the fib-ind $\begin{bmatrix} a \\ b \end{bmatrix}$. It has **depth** one. This corresponds to an mpih Betti number, namely $g_b = \text{cand}(i)$ on $(a+b)$-dimensional polytopes.

The fib-inds in $\text{cand}(i)$ with depth ≥ 1 are obtained by splitting i.

Definition The **split partial order** is generated by

$$\begin{bmatrix} \dots & a+b & \dots \\ & c+d & \end{bmatrix} >_{\text{split}} \begin{bmatrix} \dots & a \; b & \dots \\ & c \; d & \end{bmatrix}$$

provided both sides are Fibonacci indices.

We read this as saying the LHS can be **split** to give the RHS.

Lemma The mpih Betti numbers are produced by applying the split partial order to indices of the form $\begin{bmatrix} a \\ b \end{bmatrix}$.

The left partial order $\succ_{\text{left}}$

The offset = 1 indices in

$$\text{cand}\left(\begin{bmatrix}1&3\\0&1\end{bmatrix}\right)=\left\{\begin{bmatrix}1&3\\0&1\end{bmatrix},\begin{bmatrix}2&2\\0&1\end{bmatrix},\begin{bmatrix}3&1\\0&1\end{bmatrix},\quad\begin{bmatrix}1&2\\0&2\end{bmatrix}\right\}$$

are obtained by moving to the left some of the amount in the top row.

Definition The **left partial order** is generated by

$$\begin{bmatrix}\dots & a & \dots & b' & \dots\\ & c & & d & \end{bmatrix}\succ_{\text{left}}\begin{bmatrix}\dots & a' & \dots & b & \dots\\ & c & & d & \end{bmatrix}$$

provided (this is the leftward motion) $b' - b = a' - a > 0$, and also that both sides are Fibonacci indices.

Proposition For $d = 5$, the following are equivalent:

- We have $j \in \text{cand}(i)$, and j has the same offset as i.
- We can solve $i \succeq_{\text{left}} x \succeq_{\text{split}} j$ for $x \in I_d$.

Not transitive — examples

We have, using the CD notation,

$$\text{cand}(CDCC) \text{ contains } CCDC$$
$$\text{cand}(CCDC) \text{ contains } DCD$$
$$\text{cand}(CDCC) \text{ does not contain } DCD$$

and also

$$\text{cand}(DCCC) \text{ contains } CDCC$$
$$\text{cand}(CDCC) \text{ contains } CDD$$
$$\text{cand}(DCCC) \text{ does not contain } CDD$$

and thus belonging to a candidate subset is not a transitive relation.

We have to be careful when changing the offset.

The offset partial order $\succ_{\text{offset}}$

The essence of the $\succ_{\text{offset}}$ partial order is that we can increase offset, perhaps moving left as we do. Two C's combine to give a D. But there is a condition, namely not both of the C's come from the first column.

Definition The **offset partial order** is generated by

$$\begin{bmatrix} \cdots \begin{matrix} a \\ c \end{matrix} \cdots \begin{matrix} b' \\ d \end{matrix} \cdots \end{bmatrix} \succ_{\text{offset}} \begin{bmatrix} \cdots \begin{matrix} a \\ c' \end{matrix} \cdots \begin{matrix} b \\ d \end{matrix} \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots \begin{matrix} a\ b' \\ c\ d \end{matrix} \cdots \end{bmatrix} \succ_{\text{offset}} \begin{bmatrix} \cdots \begin{matrix} a\ b \\ c\ d' \end{matrix} \cdots \end{bmatrix}$$

provided (this is the increase in offset) $b' - b = c' - c = d' - d > 0$, and also that both sides are Fibonacci indices.

Note These rules give $CCDC \succ_{\text{offset}} DCD$, and $CDCC \succ_{\text{offset}} CDD$.

Note If i is depth 1 (mpih) then $i \succ_{\text{offset}} j$ has no solutions.

Note The relation $CCCD \succ_{\text{offset}} j$ has no solutions.

The candidate Betti numbers

We can now put everything together, but must take care with the order

Recall that we have Fibonacci indices W_d, and on the Fibonacci indices we have the split, left and offset partial orders. Recall also that each subset s of the Fibonacci indices produces a linear function ℓ_s on $\mathcal{F}_d$.

Definition For each i in I_d we define $s = \text{cand}(i)$ to be all j in W_d such that the equation

$$i \quad \geq_{\text{offset}} \quad x \quad \geq_{\text{left}} \quad y \quad \geq_{\text{split}} \quad j$$

can be solved, for x and y in I_d.

In other words, from i we can first increase offset some, then move left some, and then split. But the changes must be done in that order.

Definition We call this ℓ_s the ***i*-th candidate Betti number** g_i.

Lemma For $d = 5$, this definition agrees with the explicit candidate Betti numbers given earlier.

Where now? What next?

First, a loose end. For $d = 5$ we get 13 extremal s, not 8. The extra 5 may come from $X \mapsto CX$ and applying the $d = 6$ linear homology.

To make more progress towards the big goal of linear homology:

1) Test that $g_i(X) \geq 0$ for many X, eg products, joins and polars.

2) Find test polytopes in $d > 5$ that give the candidate Betti numbers. Even for $d = 6$ brute force and ignorance search is not possible.

The mpih formulas for C and I are elegant and have geometric meaning.

3) Find formulas for $g(CX)$ and $g(IX)$ in terms of $g(X)$.

4) Geometrically, what do these formulas mean?

5) For $i = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and the very special polytope, construct geometrically a vector space of the right dimension.

There are further ideas in my slides
Candidate Betti numbers for the linear homology of convex polytopes.

The End